

Building blocks of turbulence

M. Avila

Institute of Fluid Mechanics, Friedrich-Alexander-Universität Erlangen-Nürnberg, 91058 Erlangen, Germany

F. Mellibovsky

*Castelldefels School of Telecom and Aerospace Engineering (EETAC),
Universitat Politècnica de Catalunya, 08860 Barcelona, Spain*

N. Roland and B. Hof

Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany

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Although the equations governing fluid flow are well known, there are no analytical expressions that describe the complexity of turbulent motion. A recent proposition is that in analogy to low dimensional chaotic systems, turbulence is organized around unstable solutions of the governing equations which provide the building blocks of the disordered dynamics. We report the discovery of periodic solutions which just like intermittent turbulence are spatially localized and show that turbulence arises from one such solution branch.

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Fluids move in a well ordered fashion (laminar flow) when their velocity is small and in this case the flow field can usually be analytically derived from the equations of motion, the Navier–Stokes equations. However, as the inherent velocity and length scales become large, turbulence sets in and most flows of practical interest are highly disordered in space and time. One of the most extensively studied cases is the flow down a straight circular pipe, where turbulence arises despite the linear stability of the base flow [1]. Here, just like in many other wall-bounded flows turbulence first manifests itself in localised spots surrounded by laminar flow. Although experimental observations of localised turbulent structures date back to the first comprehensive investigations of turbulence [1] and their structure and kinematics have been studied extensively [2–8], a theoretical understanding is missing. More recent studies have shown that turbulent spots (called puffs in pipe flow) are generally of transient nature and that their decay is memoryless [9–11]. Nevertheless turbulence eventually becomes sustained once these structures begin to proliferate and their spreading rate outweighs their decay [12]. The Reynolds number ($Re = DU/\nu$, where D is the pipe diameter, U the mean velocity and ν the kinematic viscosity of the fluid) at which these processes balance marks a phase transition to sustained turbulence. Despite such recent advances, how these turbulent structures arise from the equations of motion is unknown.

Numerical studies of flows in short periodic domains led to the important discovery of invariant solutions of the Navier–Stokes equations featuring the main ingredients of the self-sustaining cycle of turbulent shear flows [13, 14]. In pipe flow, the simplest of these solutions are traveling waves [15, 16], satisfying

$$\mathbf{v}(x, r, \theta, t) = \mathbf{v}(x - ct, r, \theta), \quad (1)$$

where (x, r, θ) are cylindrical coordinates, t time and c the wave-speed. Traveling waves are frozen as they propagate, i.e. they are relative equilibria. Although all these exact solutions are unstable, and hence cannot be directly observed in experiments, the number of unstable directions is small, so it is expected that they play an important role in organizing the phase-space dynamics of turbulence [17–20]. As traveling waves have no dynamics but only drift in the propagation direction, more complex solutions are required to capture the properties of turbulent flows. The next level of complexity in the hierarchy of exact solutions of the governing equations is provided by relative periodic orbits (RPOs)

$$\mathbf{v}(x, r, \theta, t) = \mathbf{v}(x - \bar{c}T, r, \theta, t + T), \quad (2)$$

for which the motion appears as T -periodic in a frame co-moving at speed \bar{c} . Relative periodic orbits bifurcating from traveling waves [21] and embedded in turbulence [22] have been recently discovered in short pipes.

Although some aspects of the traveling wave solutions found in small domains, like the symmetry and the vortex streak arrangement have also been observed in turbulent pipe experiments [7, 23–25], the streamwise structure is qualitatively different. While traveling waves are streamwise periodic, with a periodicity of a few D , all turbulent structures observed close to onset are localized. Turbulent puffs have distinct laminar-turbulent interfaces characterized by a sharp velocity change at the upstream interface and a slow adjustment downstream. In this *Letter*, we present the first localised solutions that contain all spatial features of turbulent puffs and show how turbulent transients emerge from them.

Numerical simulations of pipe flow were carried out using a spectral code [26]. The computational domain was chosen to be long ($40D$) with periodic boundary conditions in the streamwise direction. In such long domains,

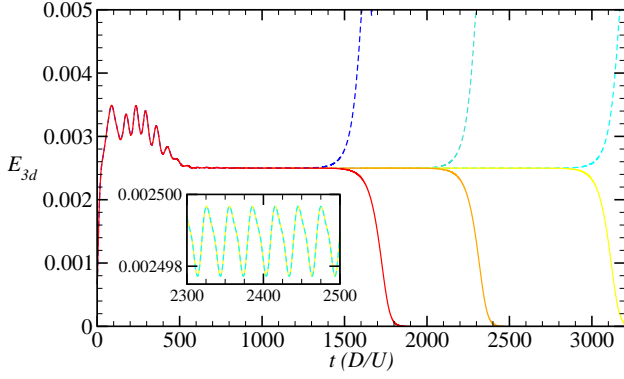


FIG. 1. Dynamics of pipe flow at the edge ($Re = 2200$). At $t = 0$ a disturbance is applied to the laminar flow and the evolution of kinetic energy (of three-dimensional Fourier modes) is subsequently monitored. The dashed lines correspond to flow trajectories that shoot up to turbulence, whereas the solid lines show trajectories that relaminarise. The edge-tracking algorithm is applied to obtain trajectories that hang around on the edge of chaos, i.e. they neither relaminarise nor go turbulent. The periodic oscillations shown in the inset (close up) suggest that trajectories on the edge are attracted to a RPO.

just like in experiments, turbulence takes the form of localised puffs and the agreement with experiments even of very subtle features like lifetime statistics is found to be excellent [11].

At first our investigation focused on the laminar-turbulent phase-space boundary by looking for initial conditions that neither turn turbulent nor relaminarise but remain in the dividing edge [27]. In long pipes the attractor in the edge (called edge state) was found to be localised but at the same time chaotic [28, 29] and the dynamics turned out to be too complex to identify underlying exact solutions. Although approaches to nearly periodic dynamics were reported in studies of symmetric invariant subspaces, in long pipes the edge state was always found to be chaotic [30]. Here we simplified the problem by restricting the dynamics subject to a π -rotational symmetry with respect to the pipe axis

$$\mathbf{v}(x, r, \theta, t) = \mathbf{v}(x, r, \theta + \pi, t) \quad (3)$$

and the reflectional symmetry

$$\mathbf{v}(x, r, \theta, t) = \mathbf{v}(x, r, -\theta, t), \quad (4)$$

which prohibits rotations about the pipe axis. Note that any solutions found in the subspace are necessarily also solutions of the full space and hence represent physical (symmetric) flow states.

The edge-tracking algorithm is as follows. First a localized disturbance is applied to the laminar flow [31] and if sufficiently strong it evolves into a turbulent puff. Subsequently, the amplitude of this puff, to which the laminar parabolic flow has been subtracted, is rescaled to obtain

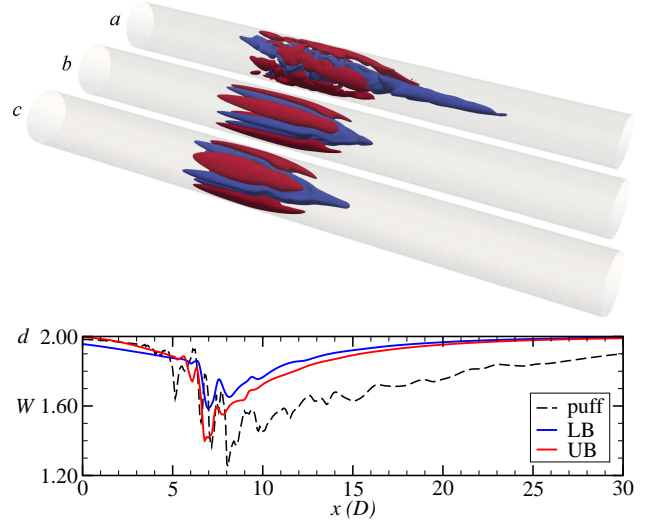


FIG. 2. (a) Turbulent puff at $Re = 1900$ and reflection-symmetric RPOs with π -rotational-symmetry at: (b) the edge (LB) at $Re = 1900$ and (c) UB at $Re = 1500$. Isosurfaces of streamwise velocity at $0.2U$ (red) and $-0.2U$ (blue) are shown. The laminar profile has been subtracted in all cases to highlight the three-dimensional structure of the flow and the views have been shrunk by a factor of 4 in the streamwise direction. $40D$ are shown out of a simulation domain of $50D$ (puff) and $40D$ (LB, UB). (d) Streamwise velocity along the pipe centerline for the structures in (a)–(c): turbulent puff (black), LB (blue) and UB (red).

a new initial condition $\mathbf{v}_\alpha = \mathbf{v}_{\text{lam}} + \alpha(\mathbf{v} - \mathbf{v}_{\text{lam}})$, where α is a constant $\alpha \in (0, 1)$, \mathbf{v} the velocity field of the puff and \mathbf{v}_{lam} the laminar flow. A simple bisection algorithm is then used to find the value of α for which the temporal evolution of \mathbf{v}_α neither relaminarizes nor goes to turbulence but remains on the edge. The procedure is illustrated in figure 1 at $Re = 2200$. After an initial transient the temporal evolution rapidly relaxes onto a periodic oscillation, suggesting that the edge state is a relative periodic orbit. Note that as time evolves new refinement bisection iterations have to be applied to keep the trajectory on the edge.

A snapshot of the edge velocity field was fed as initial guess into a purposely designed Newton–Krylov solver based on the time-stepping code [26] using standard techniques [21, 32] and rapidly converged to a RPO with period $T = 14.97 D/U$ and average drift speed $\bar{c} = 1.521 U$. Note that in order to achieve convergence we require that the residual $r = \|\mathbf{v}(T) - \mathbf{v}(0)\| < 10^{-10} \|\mathbf{v}(0)\|$, where the velocity field $\mathbf{v}(T)$ has been appropriately shifted to account for drift. Figure 1 shows that the energy oscillations have a period of $T/2$. This is due to a spatio-temporal symmetry possessed by this solution: at $t = T/2$ the velocity field is the same as $t = 0$ but reflected with respect to the plane at $\theta = 45^\circ$ (note that the plane of imposed reflection-symmetry is at $\theta = 0$). Figure 2b shows a snapshot of the RPO. The similar-

ity in the topology of its low and high velocity streaks with those of a turbulent puff (shown in 2a) is remarkable. A close inspection of the topology of streaks and vortices of this solution points at a possible connection with a rare stream-wise periodic traveling wave [33] (D2). At $Re = 2200$ this traveling wave is the edge state in short pipes [33] of length $\lambda \lesssim 5D$, whereas in the range $5 \lesssim \lambda \lesssim 10D$ the edge state is chaotic and the localized RPO is recovered as long as $\lambda \gtrsim 10D$.

As the Reynolds number is reduced the localized RPO (henceforth referred to as LB, which stands for lower branch solution) keeps fulfilling its role of separating trajectories that relaminarise from those that increase in energy towards turbulence. For $Re < 1499$ trajectories above the edge no longer result in turbulent transients but approach instead a stable (within the pi-rotational- and reflection-symmetric space) localised RPO. The visualisation of this new solution (hereafter UB, standing for upper branch solution) is shown in figure 2c and reveals a striking structural resemblance to turbulent puffs. As pointed out above, a typical signature of puffs is the sharp transition from laminar to turbulent flow at the trailing interface followed by a slow recovery towards the laminar velocity along its diffuse leading interface (see the black curve in figure 2d). This landmark of puffs is shared by LB (red curve) and UB (blue curve) and further demonstrates that the properties of localized turbulence can be captured by exact solutions of the Navier–Stokes equations.

At $Re \simeq 1428$ UB merges with LB at a saddle-node bifurcation (see figure 3a) and below this bifurcation no dynamics other than laminar are found. By continuing the UB towards larger Reynolds number we could identify a bifurcation cascade leading to turbulent transients. At $Re \simeq 1499$ the UB undergoes a Neimark–Sacker bifurcation leading to a stable 2-torus that breaks up into chaos at $Re \simeq 1504$. Although at the onset of chaos the attractor explores only a tiny portion of the phase space, this portion grows explosively as Re is increased and the chaotic attractor appears to collide with LB at $Re \simeq 1506$. This boundary crisis is likely related to the appearance of a homoclinic tangle on the edge [34, 35]. Beyond this point the attractor becomes leaky: trajectories can relaminarise after long transients. Following the ensuing chaotic saddle to larger Re confirms that turbulence in the subspace originates at this bifurcation, as illustrated in the phase-space portrait of figure 3b. We note that similar bifurcation scenarios but starting from relative equilibria have been observed in short pipes [36] and in small plane Couette cells [37], thus lacking the spatial complexity and laminar turbulent interfaces observed in practice. In these small cells a chaotic attractor emerges via period doubling bifurcations and subsequently leads to transients [37].

In summary, we have discovered exact solutions of the Navier–Stokes equations that share structure and spatial

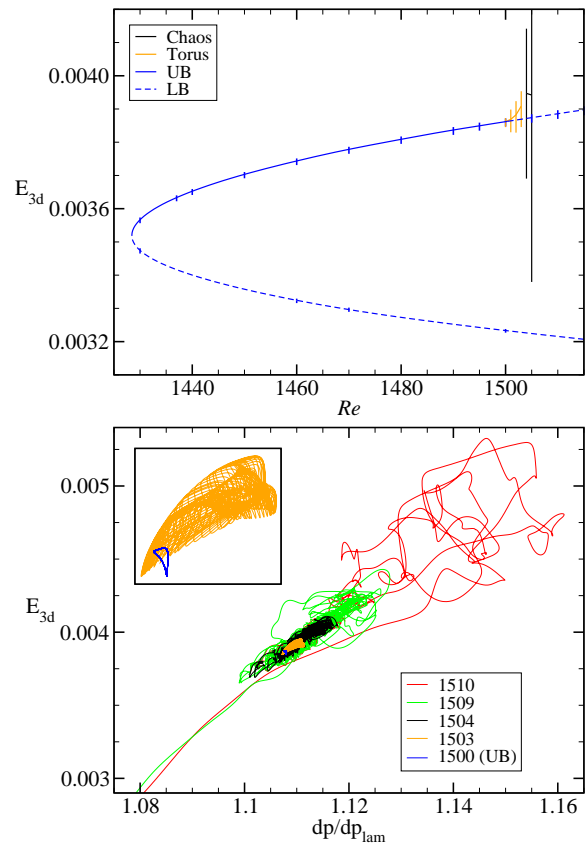


FIG. 3. (a) A saddle-node bifurcation gives rise to localised RPOs at $Re = 1428.5$: UB is stable up to $Re = 1499$, where it undergoes a supercritical Neimark-Sacker bifurcation leading to a relative 2-torus. Subsequently the torus breaks up to chaos at $Re = 1504$ and the chaos becomes transient at $Re = 1506$. The bars show the variation of energy over a period (Newton-converged LB and UB) and over very long runs (torus and chaos). LB has a single unstable direction and is the edge state: its stable manifold separates initial conditions that relaminarise uneventfully from those that shoot up (see figure 1). (b) Phase-portrait of the dynamics at several Re projected onto a two-dimensional plane defined by the energy (of three-dimensional Fourier modes) and pressure gradient required to drive a constant flow rate, normalized with the pressure gradient of laminar flow.

complexity with turbulence at onset. We have furthermore shown that a bifurcation sequence is responsible for giving rise to turbulent transients. The chaotic dynamics initially evolve on a tiny attracting region in phase-space but at slightly higher Re the attractor grows explosively, turns into a repeller and continues its rapid expansion culminating in puffs. While paving the way towards a fundamental understanding of the origin of turbulence, these findings also hint at the outstanding challenges that remain ahead, including the identification of the underlying localisation mechanisms [38, 39]. It is likely that in full space chaotic dynamics simultaneously arises from distinct localised solution branches and

that the corresponding repellers merge in global bifurcations as Re grows, thus resulting in the full complexity of turbulent motion.

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